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THE SPACE OF POSITIVE DEFINITE MATRICES AND GROMOV'S INVARIANT

BY

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ABSTRACT. The space X_d^n of $n \times n$ positive definite matrices with determinant $= 1$ is considered as a subset of $\mathbf{R}^{n(n+1)/2}$ with isometries given by $X \to AXA^t$ where the determinant of $A = 1$ and X_d^n is given its invariant Riemannian metric. This space has a collection of simplices which are preserved by the isometries and formed by projecting geometric simplices in $\mathbf{R}^{n(n+1)/2}$ to the hypersurface X_d^n. The main result of this paper is that for each n the volume of all top dimensional simplices of this type has a uniform upper bound.

This result has applications to Gromov's Invariant as defined in William P. Thurston's notes, *The geometry and topology of 3-manifolds*. The result implies that the Gromov Invariant of the fundamental class of compact manifolds which are formed as quotients of X_d^n by discrete subgroups of the isometries is nonzero. This gives the first nontrivial examples of manifolds that have a nontrivial Gromov Invariant but do not have strictly negative curvature or nonvanishing characteristic numbers.

1. Introduction. Gromov has defined an invariant on real singular homology classes which is always a pseudonorm and in some cases is a norm, cf. [**4**]. For example, if M is a closed, oriented manifold which admits a self map of degree k where $|k| > 1$, Gromov's Invariant of the fundamental class of M, denoted by $\|[M]\|$, is zero. It is interesting to find manifolds for which $\|[M]\| \neq 0$. Let X_d^n denote the set of $n \times n$ positive definite matrices with determinant $= 1$. The set of $n \times n$ matrices with determinant $= 1$, $\mathrm{SL}(n; \mathbf{R})$, acts on X_d^n by $X \to AXA^t$ where $X \in X_d^n$ and $A \in \mathrm{SL}(n; \mathbf{R})$. Give X_d^n its invariant Riemannian metric. We show that if a compact manifold M is formed as a quotient of X_d^n by a discrete group of the isometries, then Gromov's Invariant of the fundamental class of M is nonzero. The existence of compact manifolds covered by X_d^n is shown by Borel in [**1**]. Gromov has conjectured that any compact manifold M whose universal cover is a symmetric space of noncompact type has $\|[M]\| \neq 0$. X_d^n is the symmetric space for $\mathrm{SL}(n; \mathbf{R})$ so this is a special case of the conjecture. More generally, Gromov conjectured that for any manifold M with all sectional curvatures nonpositive and Ricci tensor strictly negative $\|[M]\| \neq 0$.

Thurston showed that for a compact n-manifold with all sectional curvatures satisfying $K \geq -\varepsilon$ for some $\varepsilon > 0$, $\|[M]\| \geq C(n, \varepsilon)\mathrm{vol}(M)$, where $C(n, \varepsilon)$ is a

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positive function of n and ε, cf. [8]. Also, results of Milnor in [6] and Sullivan in [7] show that if M supports an affine flat bundle of dimension n with Euler number χ then $\|[M]\| \geqslant |\chi|$, cf. [4]. The Milnor-Sullivan theorem has been generalized by Gromov to other characteristic numbers, cf. [4]. The result in this paper does not follow from either of these results. It can be checked that some sectional curvatures of X_d^n are zero so that Thurston's theorem does not apply. If M is formed as a quotient of X_d^n by a subgroup of the isometries then M can be written as X_d^n/Γ. M does support an affine flat bundle with fiber $\mathbf{R}^n$ and group $\mathrm{SL}(n;\mathbf{R})$ given by $X_d^n \times_\Gamma \mathbf{R}^n \to M$. However, it is easily checked that the only nonvanishing characteristic class for this bundle is the Euler class, which is in dimension n. If $n > 2$, the dimension of M which is $n(n+1)/2 - 1$ is greater than n. Then again the result in this paper does not follow from the Milnor-Sullivan theorem or Gromov's generalization if $n > 2$.

A simplex in X_d^n is defined to be straight if it is the projection of a geometric simplex in $\mathbf{R}^{n(n+1)/2}$ with vertices in X_d^n. In §§2 and 3 it is shown that proving $\|[M]\|$ nonzero can be reduced to showing that the volume of all top dimensional straight simplices in X_d^n has a uniform upper bound. Let X_t^n denote the space of positive definite matrices with trace $= 1$ and Riemannian metric which makes the natural map to X_d^n an isometry. Then the geometric simplices with vertices in X_d^n can be projected to X_t^n to form straight simplices in X_t^n.

The last four sections of the paper are devoted to showing that the volume of straight simplices with vertices in X_t^n has a uniform upper bound thus showing that $\|[M]\| \neq 0$ for all compact manifolds whose universal cover is X_d^n. In §4 the volume form is computed on X_t^n and X_d^n. In §5 the problem is reduced to studying straight simplices whose vertices are rank 1 matrices in the boundary of X_t^n. Also in §5 a series of theorems concerning the linear algebra of the space X_t^n are proved which are used in the estimates in the later sections. §§6 and 7 consist of a series of estimates involving the Euclidean volume of some particular slices of a straight simplex and the determinant on these slices. §8 mentions two results that follow easily from the main result of this paper. The two key results are Theorem 5.14 which gives a formula for the determinant on a simplex whose vertices are rank 1, positive semidefinite matrices in terms of the barycentric coordinates and the distinguished eigenvectors of the vertices, and Theorem 6.1 which gives an estimate on the Euclidean volume of a simplex in terms of some of these distinguished eigenvectors.

2. Gromov's Invariant. Let X be any topological space and let $C_*(X)$ denote the real singular chain complex of X. If c is any k-chain, c can be written uniquely as $c = \Sigma r_i\sigma_i$ where $r_i \in \mathbf{R}$ and σ_i is a continuous map from the standard k-simplex Δ^k into X. Define $\|c\|$ by

$$\|c\| = \sum |r_i|. \tag{2.1}$$

If α is a homology class in $H_k(X;\mathbf{R})$ define Gromov's Invariant of α by

$$\|\alpha\| = \inf\{\|c\| \mid c \text{ is a singular cycle representing } \alpha\}. \tag{2.2}$$

It is easily seen that $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$ and that $\|\lambda\alpha\| = |\lambda|\,\|\alpha\|$. However, $\|\alpha\| = 0$ need not imply that $\alpha = 0$. For example, let $X = S^n$ and let $\sigma_j\colon \Delta^n \to S^n$ be defined by $\sigma_j = f_j \circ g$ where $g\colon \Delta^n \to S^n$ is the map which identifies $\partial\Delta^n$ to a point, and $f_j\colon S^n \to S^n$ is a map of degree j. Then $\frac{1}{j}\sigma_j$ represents the fundamental class of S^n, denoted $[S^n]$. Clearly $\inf\|\frac{1}{j}\sigma_j\| = 0$, hence $\|[S^n]\| = 0$. Therefore, Gromov's Invariant is a pseudonorm on the real homology but need not be a norm. If $f\colon X \to Y$ is a continuous map,

$$\|f_*\alpha\| \leq \|\alpha\| \tag{2.3}$$

since if $\Sigma r_i\sigma_i$ represents α, $r_i(f \circ \sigma_i)$ represents $f_*\alpha$. In particular, if M and N are closed, oriented manifolds and $[M]$ and $[N]$ represent the fundamental classes we have

$$\|f_*[M]\| \leq \|[M]\|. \tag{2.4}$$

Since $f_*[M] = (\deg f)[N]$ (2.4) becomes

$$|\deg f|\,\|[N]\| \leq \|[M]\|. \tag{2.5}$$

In the case that $M = N$ and $\|[M]\| \neq 0$ we see that $|\deg f| \leq 1$. Since S^n admits self maps of all degrees, it can again be seen that $\|[S^n]\| = 0$.

Let H be a hypersurface in $\mathbf{R}^n$ with the property that if P and Q are points in H, and R is any point on the line segment $\overline{PQ}$, then the line determined by the origin and R intersects H in exactly one point. H determines a cone $C(H)$ where $C(H)$ is the set of all points P such that $\overleftrightarrow{OP}$ intersects H. There is a map $\pi\colon C(H) \to H$ given by $\pi(P) = \overleftrightarrow{OP} \cap H$. Define a singular k-simplex σ with vertices $v_0, v_1, \ldots, v_k$ in H to be straight if

$$\sigma = \pi(\text{geometric simplex spanned by } v_0, v_1, \ldots, v_k).$$

There is a function "straight" which associates to each singular k-simplex a straight k-simplex. If σ is a singular k-simplex with vertices $v_0, v_1, \ldots, v_k$ define

$$\operatorname{straight}(\sigma) = \pi(\text{geometric simplex spanned by } v_0, v_1, \ldots, v_k). \tag{2.6}$$

Give H a Riemannian metric and let G denote the group of isometries of H. Let Γ be a subgroup of G which acts properly discontinuously and freely on H. Then $M = H/\Gamma$ is a Riemannian manifold with covering space H such that the projection $p\colon H \to M$ is a local isometry.

Now assume that H has the additional property that for each singular simplex σ and each isometry g,

$$g \circ \operatorname{straight}(\sigma) = \operatorname{straight}(g \circ \sigma). \tag{2.7}$$

Let $\tau\colon \Delta^k \to M$ and let $\tilde{\tau}$ be a lift of τ to H. Define straight(τ) by

$$\operatorname{straight}(\tau) = p \circ \operatorname{straight}(\tilde{\tau}). \tag{2.8}$$

It must be shown that straight(τ) does not depend on the lift chosen. Let $\tilde{\tau}'$ be another lift of τ, and let x be a point in Δ^k. We can choose a covering transformation

g taking $\tilde{\tau}'(x)$ to $\tilde{\tau}(x)$. It is easily seen that g is an isometry of H. We have

$$\begin{aligned} p \circ \operatorname{straight}(\tilde{\tau})(x) &= p \circ \operatorname{straight}(g \circ \tilde{\tau}')(x) \\ &= p \circ g \circ \operatorname{straight}(\tilde{\tau}')(x) \\ &= p \circ \operatorname{straight}(\tilde{\tau}')(x). \end{aligned}$$

Hence staight(τ) is independent of the lift. Straight extends linearly to a map $C_*(M) \to C_*(M)$ which is clearly a chain map. If σ is a singular simplex in H there is a canonical homotopy h_σ of σ with straight(σ) where h_σ is the projection to H of the linear homotopy between σ and straight(σ) in $\mathbf{R}^n$. That is,

$$h_\sigma(x,t) = \pi[(1-t)\cdot\sigma(x) + t\cdot \operatorname{straight}(\sigma)(x)]. \tag{2.9}$$

Then define D: $C_k(M) \to C_{k+1}(M)$ in the standard fashion by first defining $D(\sigma)$ as

$$D(\sigma) = \sum_{i=0}^{k} (-1)^i p \circ h_{\tilde{\sigma}} \mid (v_0, v_1, \ldots, v_i, w_i, \ldots, w_k), \tag{2.10}$$

where $v_0, v_1, \ldots, v_k$ are the vertices of $\Delta^k \times 0$, $w_0, w_1, \ldots, w_k$ are the vertices of $\Delta^k \times 1$, and $(v_0, v_1, \ldots, v_i, w_i, \ldots, w_k)$ denotes the $k+1$ simplex spanned by the indicated set of $k+2$ points, cf. e.g. [**3**]. By extending D linearly we get a map taking $C_k(M)$ into $C_{k+1}(M)$. By a standard argument it is checked that D is a chain homotopy between straight and the identity, cf. [**3**]. Hence τ and straight(τ) represent the same homology class. Clearly $\|\operatorname{straight}(c)\| \leqslant \|c\|$ for all chains c. Therefore, to compute Gromov's Invariant of a homology class it suffices to consider only straight simplices.

THEOREM. *If the volume of every straight $n-1$ simplex in H is bounded above by a positive constant v_n, then $\|[M]\| \geqslant \operatorname{vol}(M)/v_n$ for every closed oriented manifold $M = H/\Gamma$.*

PROOF. Let dV denote the volume form of M, and let $\Sigma r_i\sigma_i$ be any straight cycle representing $[M]$. Then

$$\operatorname{vol}(M) = \int_{\Sigma r_i\sigma_i} dV = \sum r_i \int_{\Delta^{n-1}} \sigma_i^* \, dV \leqslant \sum |r_i| \left| \int_{\Delta^{n-1}} \sigma_i^* \, dV \right| \leqslant v_n\left(\sum |r_i|\right).$$

The last inequality holds since M is locally isometric with H. Dividing we have

$$\sum |r_i| \geqslant \frac{\operatorname{vol}(M)}{v_n}. \tag{2.11}$$

Then taking the infimum over all straight cycles representing $[M]$ we have

$$\|[M]\| \geqslant \frac{\operatorname{vol}(M)}{v_n}. \tag{2.12}$$

For example, if H is the hyperboloid model of hyperbolic 2-space the Gauss-Bonnet Formula shows that $v_n = \pi$. Hence $\|[M]\|$ is nonzero for every closed, oriented hyperbolic 2-manifold. More generally it can be shown that for hyperbolic n-space $v_n \leqslant \pi/(n-1)!$ so that any closed, oriented hyperbolic n-manifold has $\|[M]\| \neq 0$.

By a theorem due to Gromov, if M is a closed, oriented manifold with universal cover E

$$\|[M]\| = C\,\mathrm{vol}(M), \tag{2.13}$$

where C depends only upon E, cf. [**4**]. Gromov also proves that if E is hyperbolic n-space $C = 1/v_n$ so that the inequality (2.12) is actually equality, cf. [**8**].

A pseudonorm can also be defined on the real cochain complex $C^*(X)$. If c is a k-cochain define

$$\|c\|_\infty = \sup_\sigma |c(\sigma)|, \tag{2.14}$$

where σ ranges over all singular simplices $\sigma\colon \Delta^k \to X$. If $\|c\|_\infty < \infty$ the cochain is said to be bounded. If $\alpha \in H^k(X; \mathbf{R})$ define $\|\alpha\|_\infty$ by

$$\|\alpha\|_\infty = \inf\{\|c\|_\infty \mid c \text{ is a cochain representing } \alpha\}. \tag{2.15}$$

Let $\zeta \in H^n(M; \mathbf{R})$ be the fundamental class. Gromov shows that $\|\zeta\|_\infty < \infty$ if and only if $\|[M]\| > 0$, cf. [**4**].

3. The space of positive definite matrices. Let X^n denote the set of positive definite $n \times n$ matrices. A matrix X is defined to be positive definite if $X^t = X$ and $\langle Xv, v\rangle > 0$ for all nonzero vectors v in $\mathbf{R}^n$. Equivalently, X is positive definite if and only if it admits an orthonormal basis of eigenvectors all of which have positive eigenvalues. Also equivalent, X is positive definite if and only if there is a nonsingular matrix B such that $X = BB^t$. X^n is naturally a subset of $\mathbf{R}^{n(n+1)/2}$ with coordinates given by (x_{ij}) such that $x_{ij} = x_{ji}$. X^n is clearly an open subset of $\mathbf{R}^{n(n+1)/2}$ and it is also convex since if P and Q are points in X^n

$$\langle (1-t)P + tQ(v), v\rangle = (1-t)\langle Pv, v\rangle + t\langle Qv, v\rangle > 0 \tag{3.1}$$

for all t such that $0 \leqslant t \leqslant 1$. Let $\mathrm{Cl}(X^n)$ denote the closure of X^n in $\mathbf{R}^{n(n+1)/2}$. $\mathrm{Cl}(X^n)$ is the set of all symmetric matrices X with $\langle Xv, v\rangle \geqslant 0$ for all vectors v in $\mathbf{R}^n$. Equivalently, $X \in \mathrm{Cl}(X^n)$ if X admits an orthonormal basis of eigenvectors all of which have nonnegative eigenvalues, or if $X = BB^t$ for some matrix B. $\mathrm{Cl}(X^n)$ is clearly also convex.

If $A \in \mathrm{GL}(n; \mathbf{R})$ and $X \in X^n$ then $AXA^t \in X^n$ since if $X = BB^t$, $AXA^t = AB(AB)^t$ which is then positive definite. Therefore, $\mathrm{GL}(n; \mathbf{R})$ acts on X^n by $A^*X = AXA^t$. It is easily seen that the Riemannian metric defined by $\mathrm{trace}(X^{-1}dX\,X^{-1}dX)$ is invariant under the action of $\mathrm{GL}(n; \mathbf{R})$, hence the maps $X \to AXA^t$ are isometries of X^n. This is the unique metric up to a constant factor which is invariant under $\mathrm{GL}(n; \mathbf{R})$. It can be shown that all the isometries of X^n are of this form. The map $X \to AXA^t$ will be denoted by g_A.

Let X^n_d denote the subset of X^n defined by determinant $(X) = 1$. $\mathrm{SL}(n; \mathbf{R})$ acts on X^n_d by $A^*X = AXA^t$. The restriction of the Riemannian metric of X^n gives a Riemannian metric on X^n_d for which the maps g_A are isometries whenever $A \in \mathrm{SL}(n; \mathbf{R})$. There is a map $\pi_d\colon X^n \to X^n_d$ defined by

$$\pi_d(x_{ij}) = \left[\frac{x_{ij}}{(\det X)^{1/n}}\right]. \tag{3.2}$$

If $X \in X^n$ there is a line determined by X and the origin parametrized by tX. Clearly this line intersects X^n_d in precisely one point and the map π_d is the projection of X along this line to the hypersurface X^n_d. It is also easily seen that the cone determined by X^n_d is precisely X^n. Straight simplices can then be defined for X^n_d as in §2.

THEOREM 3.3. straight$(g_A(\sigma)) = g_A($straight$(\sigma))$ *if* $A \in \mathrm{SL}(n; \mathbf{R})$.

PROOF. It is immediate that $\pi_d \circ g_A = g_A \circ \pi_d$. Suppose first that σ is a straight simplex in X^n_d. Then there is a geometric simplex σ' in X^n such that $\pi_d(\sigma') = \sigma$. $g_A(\sigma)$ is also a straight simplex since $\pi_d(g_A(\sigma')) = g_A(\pi_d(\sigma')) = g_A(\sigma)$. Now if σ is any k-simplex in X^n_d, straight$(g_A(\sigma))$ and $g_A($straight$(\sigma))$ are both straight simplices with the same set of vertices and hence are equal.

Then by the discussion in §2, if $M = X^n_d/\Gamma$ we may compute Gromov's Invariant for homology classes of M by considering only straight simplices. Another description of X^n_d is given by letting the points be the set of lines through the origin which intersect X^n_d. More generally, this model can be obtained for any hypersurface H which satisfies the conditions given in §2.

Let X^n_t denote the subset of X^n defined by trace$(X) = 1$. By pulling back the Riemannian metric of X^n_d by the map π_d it can be shown, but is not needed here, that the metric of X^n_t which makes X^n_d and X^n_t isometric is $\operatorname{trace}(X^{-1}dX\,X^{-1}dX) - \frac{1}{n}\operatorname{trace}(X^{-1}dX)^2$. Since X^n_t is a hyperplane in $\mathbf{R}^{n(n+1)/2}$ defined by $\sum_{i=1}^n x_{ii} = 1$ the tangent space at each point may be identified with the symmetric matrices with trace $= 0$. Define a map $\pi_t\colon X^n \to X^n_t$ by

$$\pi_t(x_{ij}) = \left[\frac{x_{ij}}{\operatorname{trace}(X)}\right]. \tag{3.4}$$

The isometries of X^n_t are given by $X \to \pi_t \circ g_A \circ \pi_d(X)$ where $A \in \mathrm{SL}(n; \mathbf{R})$ or equivalently, as can be easily checked, by $X \to \pi_t \circ g_A(X)$ where $A \in \mathrm{GL}(n; \mathbf{R})$. In order to simplify notation, define $\delta(i)$ by $\delta(i) = i(i+1)/2 - 1$. Then X^n_d and X^n_t are $\delta(n)$ manifolds.

The main result in this paper is that the volume of any straight simplex in X^n_d is bounded above by a function $C(n)$ which depends only on n. This then shows that if $M = X^n_d/\Gamma$ is closed and orientable,

$$\|[M]\| \geqslant \frac{\operatorname{vol}(M)}{C(n)} > 0. \tag{3.5}$$

Equivalent to showing that $\operatorname{vol}(\sigma) \leqslant C(n)$ for any straight simplex is showing that $\operatorname{vol}(\pi_t(\sigma)) \leqslant C(n)$ since π_t is an isometry from X^n_d to X^n_t. If σ is a straight simplex with vertices $P_0, P_1, \ldots, P_{\delta(n)}$, $\pi_t(\sigma)$ is the intersection of the hyperplane X^n_t with the $\delta(n)+1$ geometric simplex spanned by $0, P_0, P_1, \ldots, P_{\delta(n)}$ and is therefore a geometric simplex in the hyperplane.

For example, X^2 is the set

$$\{(x_{11}, x_{22}, x_{21}) \in \mathbf{R}^3 \mid x_{11}x_{22} - x_{21}^2 > 0,\ x_{11} > 0, \text{ and } x_{22} > 0\}.$$

This is seen to be the interior of a circular cone in $\mathbf{R}^3$. The boundary of X^2 is the cone satisfying $x_{11}x_{22} - x_{21}^2 = 0$, and $x_{11}, x_{22} > 0$. The surface X^2_d is one sheet of the hyperboloid defined by $x_{11}x_{22} - x_{21}^2 = 1$. X_d^2 is easily seen to be isometric to hyperbolic 2-space H^2 by using the hyperbolic model for H^2. Here the straight 2-simplices in X_d^2 are precisely the geodesic triangles of H^2. It follows from the Gauss-Bonnet Formula that the area of any geodesic triangle is π − (sum of the interior angles). Hence $C(2)$ may be taken to be π. X_t^2 is the set satisfying $x_{11}(1 - x_{11}) - x_{21}^2 > 0$ and $x_{11} > 0$, or equivalently $(x_{11} - \frac{1}{2})^2 + x_{21}^2 < (\frac{1}{2})^2$. Therefore, X_t^2 is the interior of a disk of radius $\frac{1}{2}$ centered at $(x_{11}, x_{21}) = (\frac{1}{2}, 0)$ and the boundary of X_t^2 is the circle $(x_{11} - \frac{1}{2})^2 + x_{21}^2 = (\frac{1}{2})^2$. X_t^2 is actually the Klein model of hyperbolic space. The simplices $\pi_t(\sigma)$ are the geometric simplices spanned by the projection of the vertices of σ. If τ is any geometric simplex in X_t^2 then there is a simplex with vertices on the boundary which contains τ. The area of a simplex with vertices on the boundary is π so again we see that $C(2)$ can be taken to be π.

Now let $n > 2$. Given a geometric simplex in X_t^n it will be shown that it can be covered by a set of simplices (whose cardinality depends only on n), each of which has as its vertices rank 1, positive semidefinite matrices. It is then shown that any simplex of this type has volume which is bounded by a function of n.

4. The volume form. The volume form is computed on X_d^n and X_t^n by finding a $\delta(n)$ form on X_d^n which is invariant under the isometries of X_d^n. Up to a constant factor, this form is the volume form on X_d^n. It is pulled back to X_t^n to get the volume form there. The notation a_{ij} represents the entry of a matrix in the ith row and jth column. The notation

$$\begin{array}{cccc} 1, k_1 & k_1+1, k_2 & k_2+1, k_3 & \\ \left[a_{ij}\right. & |\quad b_{ij} & |\quad c_{ij} & \left.|\ \cdots\right] \end{array}$$

represents the matrix whose entries in the first k_1 column are given by the a_{ij}, and whose entries in the columns $k_1 + 1$ through k_2 are given by b_{ij}, and so on. If $\omega_1, \omega_2, \ldots, \omega_q$ are forms, the symbol $\bigwedge_{r \neq i} \omega_r$ denotes the product $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \hat{\omega}_i \wedge \cdots \wedge \omega_q$.

THEOREM 4.1. *Let $x_1, x_2, \ldots, x_{\delta(n)+1}$ be an ordering of the coordinates on X^n, x_{ij}. Define a $\delta(n)$ form ω on X^n by*

$$\omega = \sum_{i=1}^{\delta(n)+1} (-1)^{i+1} x_i \bigwedge_{r \neq i} dx_r.$$

Let $A \in \mathrm{SL}(n; \mathbf{R})$ and g_A the associated map of X^n. Then $g_A^* \omega = \omega$.

PROOF. Define a $\delta(n) + 1$ form η on X^n by $\eta = \bigwedge_{i=1}^{\delta(n)+1} dx_i$. Then

$$g_A^* \eta = \det\left[\frac{\partial (g_A)_i}{\partial x_j}\right] \eta. \tag{4.2}$$

Denote $\det[\partial(g_A)_i / \partial x_j]$ by $h(A)$, and let $B \in \mathrm{SL}(n; \mathbf{R})$. Then

$$(g_B \circ g_A)^* \eta = g_A^*(h(B)\eta) = h(A)h(B)\eta.$$

Also

$$g_B \circ g_A(X) = B(AXA^t)B^t = (BA)X(BA)^t,$$

so $(g_B \circ g_A)^*\eta = h(BA)\eta$. Then we get $h(BA) = h(B)h(A)$. Since h clearly varies smoothly in $\mathrm{SL}(n; \mathbf{R})$, h is a Lie group homomorphism from $\mathrm{SL}(n; \mathbf{R})$ into $\mathbf{R}^*$. But $\mathrm{SL}(n; \mathbf{R})$ is a simple Lie group and therefore the kernel of h must be $\mathrm{SL}(n; \mathbf{R})$. Hence (4.2) becomes $g_A^*\eta = \eta$.

The action of $\mathrm{SL}(n; \mathbf{R})$ on X^n is the restriction of a linear action of $\mathrm{SL}(n; \mathbf{R})$ on the vector space of symmetric matrices, hence the radial vector field $R = \Sigma x_i\, \partial/\partial x_i$ is also invariant under g_A. Then $\omega = R \lrcorner \eta$ is invariant under g_A.

Now restrict ω to a form ω_d on X^n_d. ω_d is invariant under the isometries of X^n_d. Since $\mathrm{SL}(n; \mathbf{R})$ acts transitively on X^n_d, ω must be a constant multiple of the volume form on X^n_d.

Theorem 4.3. *The volume form on X^n_t is $1/(\det X)^{(n+1)/2} \bigwedge_{r \neq n} dx_r$ up to a constant factor where the first n coordinates in the listing $x_1, x_2, \ldots, x_{\delta(n)+1}$ are defined by $x_i = x_{ii}$.*

Proof. Up to a constant factor the volume form on X^n_t is $\pi_d^*\omega_d$. The coefficient of $\bigwedge_{r \neq k} dx_r$ in the term corresponding to x_i is

$$(-1)^{i+1}\frac{x_i}{(\det X)^{1/n}}\left(\text{minor of } \frac{\partial(\pi_d)_i}{\partial x_k} \text{ in the Jacobian matrix of } \pi_d\right).$$

The entire coefficient of $\bigwedge_{r \neq k} dx_r$ is then

$$\det\begin{array}{c} \begin{array}{ccc} 1 & 2,k & k+1, \partial(n)+1 \end{array} \\ \left[\begin{array}{c|c|c} \dfrac{x_i}{(\det X)^{1/n}} & \dfrac{\partial(\pi_d)_i}{\partial x_{j-1}} & \dfrac{\partial(\pi_d)_i}{\partial x_j} \end{array}\right] \end{array}.$$

We have

$$\frac{\partial(\pi_d)_i}{\partial x_j} = \frac{(\det X)^{1/n}\delta_i^j - x_i\dfrac{\partial}{\partial x_j}(\det X)^{1/n}}{(\det X)^{2/n}}.$$

For each $j \neq k$ multiply the first column by $(\partial/\partial x_j)(\det X)^{1/n}/(\det X)^{1/n}$ and add to the column corresponding to $\partial/\partial x_j$. This yields

$$\det\begin{array}{c} \begin{array}{ccc} 1 & 2,k & k+1, \delta(n)+1 \end{array} \\ \left[\begin{array}{c|c|c} \dfrac{x_i}{(\det X)^{1/n}} & \dfrac{\delta^i_{j-1}}{(\det X)^{1/n}} & \dfrac{\delta^i_j}{(\det X)^{1/n}} \end{array}\right] \end{array}$$

$$= (-1)^{k-1}\frac{x_k}{\left[(\det X)^{1/n}\right]^{n(n+1)/2}}.$$

Then

$$\pi_d^*\omega_d = \sum_{i=1}^{\delta(n)+1}(-1)^{i+1}\frac{x_i}{(\det X)^{(n+1)/2}}\bigwedge_{r \neq i} dx_r. \tag{4.4}$$

This form is the pullback of ω_d to the cone X^n. On the hyperplane X_t^n, $\Sigma_{i=1}^n x_i = 1$ so that $\Sigma_{i=1}^n dx_i = 0$. Then the terms of $\pi_d^* \omega_d$ which involve all of $dx_1, dx_2, \ldots, dx_n$ must vanish. Then $\pi_d^* \omega_d$ becomes

$$\sum_{i=1}^{n} (-1)^{i+1} \frac{x_i}{(\det X)^{(n+1)/2}} \bigwedge_{r \neq i} dx_r. \tag{4.5}$$

In each term substitute $dx_n = -\Sigma_{i=1}^{n-1} dx_i$. Then (4.5) becomes

$$\begin{aligned} &\sum_{i=1}^{n} (-1)^{i+1} \frac{x_i}{(\det X)^{(n+1)/2}} \left(\bigwedge_{\substack{r=1 \\ r \neq i}}^{n-1} dx_r \right) \wedge (-dx_i) \wedge \left(\bigwedge_{r=n+1}^{\delta(n+1)} dx_r \right) \\ &\qquad = \sum_{i=1}^{n} (-1)^{i+1} \frac{(-1)^{n-i-1}}{(\det X)^{(n+1)/2}} \bigwedge_{r \neq n} dx_r. \end{aligned}$$

Therefore, up to a constant factor

$$\frac{1}{(\det X)^{(n+1)/2}} \bigwedge_{r \neq n} dx_r \tag{4.6}$$

is the volume form on X_t^n. It may be checked that with an appropriate choice of orientation the volume form on X_t^n is

$$(2n)^{n(n-1)/4} \frac{1}{(\det X)^{(n+1)/2}} \bigwedge_{r \neq n} dx_r. \tag{4.7}$$

The constant $(2n)^{n(n-1)/4}$ is denoted by $C_0(n)$.

5. Linear algebra and the space of positive definite matrices. It will be shown that finding a function of n, $C(n)$, with $\mathrm{vol}(\sigma) \leqslant C(n)$ for all straight $\delta(n)$ simplices σ in X_t^n can be reduced to the case where all vertices of σ are rank 1, by using Carathéodory's Theorem. The problem can be further reduced to showing that for each straight simplex the volume of the subset with barycentric coordinates satisfying $t_i \geqslant t_{i+1}$ for all i has volume bounded above by a function of n. The notation $(X_t^n)_1$ will be used for the subset of $\mathrm{Cl}(X_t^n)$ consisting of rank 1 matrices.

THEOREM 5.1. *Let $P_0 \in \mathrm{Cl}(X_t^n)$ with $\mathrm{rank}(P_0) = k > 1$. Let $v_1, v_2, \ldots, v_k$ denote unit length eigenvectors of P_0 which have nonzero eigenvalues and let λ denote the eigenvalue corresponding to v_1. Let $P_1 \in (X_t^n)_1$ with unit length eigenvector v_1. Define $P(t)$ by $P(t) = (1-t)P_0 + tP_1$. Then $P(t)$ is positive semidefinite in the interval $[\alpha, 1]$ where $\alpha = \lambda/(\lambda - 1) < 0$ and $P(\alpha)$ has rank $k-1$.*

PROOF. $\langle P(t)v, v\rangle = (1-t)\langle P_0 v, v\rangle + t\langle P_1 v, v\rangle$. If $t > 1$ this sum is clearly negative for some vectors in $v_1^{\perp}$. Since $P_1 v_1 = v_1$,

$$\langle P(t)v_1, v_1\rangle = (1-t)\lambda + t.$$

$\langle P(t)v_1, v_1\rangle \geqslant 0$ for t in $[\lambda/(\lambda-1), 1]$ and is 0 when $t = \lambda/(\lambda - 1)$. Then $\langle P(t)v, v\rangle \geqslant 0$ for all v if t is in $[\lambda/(\lambda-1), 1]$ since it is nonnegative on an orthonormal basis. The rank of $P(\lambda/(\lambda-1))$ is $k-1$ because the only eigenvectors of $P(\lambda/(\lambda-1))$ which have nonzero eigenvalues are $v_2, v_3, \ldots, v_k$.

CARATHÉODORY'S THEOREM. *If X is any point of the convex hull of a closed bounded set S in n-space, there is a set of q points $P = \{P_1, P_2, \ldots, P_q\} P \subset S$, $q \leq n+1$, such that X belongs to the convex hull of P, cf.* **[2]**.

DEFINITION 5.2. If $P_0, P_1, \ldots, P_{\delta(n)}$ are vertices of any straight simplex σ in $\mathrm{Cl}(X_t^n)$ define the rank of σ, denoted $R(\sigma)$, by $R(\sigma) = \sum_{i=0}^{\delta(n)} \mathrm{rank}(P_i)$.

THEOREM 5.3. *Every straight $\delta(n)$ simplex in $\mathrm{Cl}(X_t^n)$ can be covered by a set of simplices with vertices in $(X_t^n)_1$ of cardinality $\leq [n(n+1)/2+1]^{(n+1)n(n-1)/2}$.*

PROOF. Let $P_0, P_1, \ldots, P_{\delta(n)}$ be vertices of a straight simplex σ in $\mathrm{Cl}(X_t^n)$. If $\mathrm{rank}(P_i) > 1$ for some i, draw a line from P_i to a rank 1 matrix Q where the eigenvector of Q which has a positive eigenvalue is also an eigenvector of P_i with a corresponding positive eigenvalue. By Theorem 5.1 this line may be extended in the direction from Q to P_i to a matrix R. By Theorem 5.1 $\mathrm{rank}(R) = \mathrm{rank}(P_i) - 1$. Let $S = \{P_0, P_1, \ldots, \hat{P}_i, \ldots, P_{\delta(n)}, Q, R\}$. By the construction, each point X of σ is in the convex hull of S since each vertex is. Then by Carathéodory's Theorem, X is in at least one of the simplices formed from S by deleting one point and taking the convex hull of the remaining ones. Therefore, σ is covered by this set of no more than $n(n+1)/2+1$ simplices. Each of the new simplices has rank which is no more than $R(\sigma) - 1$. Now repeat the process with each new simplex to form a set of no more than $(n(n+1)/2+1)^2$ simplices, each of which has rank no more than $R(\sigma) - 2$. For any straight $\delta(n)$ simplex τ, $R(\tau) \leq n^2(n+1)/2$ while if the vertices of τ are points in $(X_t^n)_1$, $R(\tau) = n(n+1)/2$. Therefore, the process described above need be repeated no more than

$$\frac{n^2(n+1)}{2} - \frac{n(n+1)}{2} = \frac{(n+1)n(n-1)}{2}$$

times until all of the new simplices formed have vertices in $(X_t^n)_1$. The total number of simplices necessary is then bounded above by

$$\left[\frac{n(n+1)}{2} + 1\right]^{(n+1)n(n-1)/2}.$$

The problem has then been reduced to considering only those straight simplices with vertices in $(X_t^n)_1$.

THEOREM 5.4. *If for each straight $\delta(n)$ simplex with vertices in $(X_t^n)_1$ the subset defined by those points whose barycentric coordinates satisfy $t_i \geq t_{i+1}$ for all i has volume bounded by a function depending only on n, then the volume of each straight $\delta(n)$ simplex in $\mathrm{Cl}(X_t^n)$ has volume bounded by a function of n.*

PROOF. Let σ be any straight $\delta(n)$ simplex in $\mathrm{Cl}(X_t^n)$. By Theorem 5.3 it suffices to assume that all vertices of σ, $P_0, P_1, \ldots, P_{\delta(n)}$, are in $(X_t^n)_1$. Let τ be a permutation of $\{0, 1, \ldots, \delta(n)\}$. Define S_τ by

$$S_\tau = \left\{ \sum_{i=0}^{\delta(n)} t_i P_{\tau(i)} \,\middle|\, t_i \geq t_{i+1} \text{ for all } i \right\}.$$

Then $\bigcup_\tau S_\tau = \sigma$ and therefore the theorem is established.

The coordinates on σ given by

$$\sum_{i=0}^{\delta(n)} u_i \prod_{k<i} (1-u_k) P_i, \tag{5.5}$$

where $u_{\delta(n)} = 1 - u_{\delta(n)-1}$ and $0 \leqslant u_i \leqslant 1$ for all i give an alternate set of coordinates on σ. If some $u_k = 1$ define $u_j = 1$ if $j > k$. Barycentric coordinates on a simplex measure distances from a vertex toward the face opposite that vertex. In contrast, $u_i = 1$ for points on the simplex spanned by $P_0, P_1, \ldots, P_i$ while $u_i = 0$ on the face opposite P_i. The u coordinates are measuring the distances from the subsimplices spanned by the first i vertices toward the face opposite P_i.

THEOREM 5.6. *Let σ be a straight simplex with vertices $P_0, P_1, \ldots, P_{\delta(n)}$ each of which is in $(X_t^n)_1$. On the subset of σ defined by $t_i \geqslant t_{i+1}$ for all i each $u_j \geqslant 1/(\delta(n)+1)$.*

PROOF. We may rearrange the coordinates on σ by

$$\sum_{i=0}^{j-1} u_i \left[\prod_{k<i} (1-u_k) \right] P_i + \prod_{k=0}^{j-1} (1-u_k) \left[\sum_{i=j}^{\delta(n)} u_i \prod_{k=j}^{i-1} (1-u_k) \right] P_i.$$

By the assumption that the t_i decrease we have

$$u_j \geqslant u_{j+1}(1-u_j) \geqslant u_{j+2}(1-u_{j+1})(1-u_j) \geqslant \cdots.$$

But these are coordinates on the face of σ spanned by $P_j, P_{j+1}, \ldots, P_{\delta(n)}$ and therefore their sum is 1. Then

$$u_j \geqslant \frac{1}{\delta(n)+1-j} \geqslant \frac{1}{\delta(n)+1}.$$

The structure on the boundary of X_t^n plays an important part in getting the uniform bound on the volume of simplices with vertices in $(X_t^n)_1$. For $n \geqslant 3$ the boundary consists of matrices of different ranks since there are boundary matrices of all ranks from 1 to $n-1$. It is easily seen that the subset of $\mathrm{Cl}(X_t^n)$ consisting of those matrices with rank $\geqslant k$ is an open subset of $\mathrm{Cl}(X_t^n)$. In contrast to the case $n = 2$, subsets of a straight simplex other than the vertices can lie on the boundary if $n \geqslant 3$. For example, if P and Q are rank 1 matrices the interior of the line segment connecting them always consists of rank 2 matrices. In general, the rank of matrices in the interior of a simplex is at least as large as the rank of any matrix on the boundary of the simplex but need not be strictly larger. In constructing the estimates on the volume of a simplex, it will be necessary to know what dimensional simplices can lie in the subset of the boundary of X_t^n which consists of those matrices with rank $\leqslant k$ for each k. This subset will be denoted by $(X_t^n)_k$. This is worked out in Theorem 5.7.

THEOREM 5.7. *The maximum dimensional simplex which can lie in $(X_t^n)_k$ is $\delta(k)$.*

PROOF. Let $P_0, P_1, \ldots, P_m$ be vertices of a straight simplex $\sigma \subset (X_t^n)_k$. If $P = \sum_{i=0}^m t_i P_i$ is an interior point of σ and v is in the null space N of P we have

$$0 = \langle Pv, v \rangle = \sum_{i=0}^{m} t_i \langle P_i v, v \rangle$$

so $P_i v = 0$ for each i. Let $v_1, v_2, \ldots, v_j$ be an orthonormal basis for $N^{\perp}$ and complete this to an orthonormal basis of $\mathbf{R}^n$. Note that $j \leqslant k$. Let A be the matrix whose rows are the vectors v_i. Then $\pi_t(g_A(P_i))$ is of the form

$$\begin{array}{cc} & k \\ k & \left[\begin{array}{c|c} \ast & 0 \\ \hline 0 & 0 \end{array}\right] \end{array}. \tag{5.8}$$

Let Q_i be the upper $k \times k$ submatrix of P_i. Each $Q_i - Q_0$ for $1 \leqslant i \leqslant m$ represents a vector in $\mathbf{R}^{\delta(k)}$. If σ is nondegenerate these vectors must be independent. Then we must have $m \leqslant \delta(k)$ as claimed.

If $P \in (X_t^n)_1$ then P has only one eigenvector v with a corresponding positive eigenvalue (which must equal 1). If $e_1, e_2, \ldots, e_n$ represent the standard basis of $\mathbf{R}^n$, it is easy to see that

$$P = [v \cdot e_1 v \cdot e_2 \cdots v \cdot e_n]^t [v \cdot e_1 v \cdot e_2 \cdots v \cdot e_n]. \tag{5.9}$$

Up to a sign v is uniquely determined by P.

DEFINITION 5.10. A vector v will be called the distinguished eigenvector of a matrix P in $(X_t^n)_1$ if v is a unit eigenvector, and if $v \cdot e_j > 0$ and $v \cdot e_{j+1} = v \cdot e_{j+2} = \cdots = v \cdot e_n = 0$ for some j, $1 \leqslant j \leqslant n$. The distinguished eigenvector of P_i will be denoted by v_i.

Note that a straight simplex with vertices in $(X_t^n)_1$ lies in $(X_t^n)_k$ if and only if the corresponding distinguished eigenvectors lie in a k-plane in $\mathbf{R}^n$. The next two theorems will be used to put an arbitrary straight simplex with vertices in $(X_t^n)_1$ into a suitable general position for estimating its volume.

THEOREM 5.11. *Define a map* $\alpha\colon (X_t^n)_1 \to S^{n-1}$ *by* $\alpha(P) = v$ *where* v *is the distinguished eigenvector of* P. *Define a function* $\beta\colon \mathbf{R}^n - 0 \to S^{n-1}$ *which takes each nonzero vector* v *in* $\mathbf{R}^n$ *to a unit vector in the direction of* v *whose last nonzero component is positive* (β *is not continuous*). *Let* $A \in \mathrm{GL}(n;\mathbf{R})$. *Then the following diagram commutes.*

$$\begin{tikzcd} (X_t^n)_1 \arrow[r, "\pi_t \circ g_A"] \arrow[d, "\alpha"'] & (X_t^n)_1 \arrow[d, "\alpha"] \\ S^{n-1} \arrow[r, "\beta \circ A"] & S^{n-1} \end{tikzcd}$$

PROOF. Let $P \in (X_t^n)_1$ with distinguished eigenvector v. Consider v as a column vector. Av is an eigenvector of APA^t because

$$(APA^t)(Av) = Avv^tA^tAv = \langle Av, Av\rangle Av. \tag{5.12}$$

The eigenvalue is nonzero since $A \in \mathrm{GL}(n;\mathbf{R})$. Then $\alpha(\pi_t(APA^t))$ is the multiple of Av with unit length and last nonzero entry positive. $\beta(A(\alpha(P))) = \beta(Av)$ is by definition the same quantity so the diagram commutes.

THEOREM 5.13. *Let $P_1, P_2, \ldots, P_n$ be points in $(X_t^n)_1$, such that the distinguished eigenvectors of the P_i form a basis for $\mathbf{R}^n$. Let $P_{n+1} \neq P_n$ be in $(X_t^n)_1$ such that its distinguished eigenvector does not lie in the hyperplane spanned by $v_1, v_2, \ldots, v_{n-1}$. Then there is an isometry h of X_t^n such that $h(P_i) = E_i$ for $1 \leqslant i \leqslant n$ where E_i is the matrix with entry $a_{ii} = 1$ and other entries 0, and $h(P_{n+1})$ is a matrix whose entry $a_{nn} = \frac{1}{2}$.*

PROOF. Choose $A \in \mathrm{GL}(n; \mathbf{R})$ such that $Av_i = e_i$ for $1 \leqslant i \leqslant n$. Then by Theorem 5.11, $\pi_t(g_A(P_i))$ is the matrix whose distinguished eigenvector is e_i, hence $\pi_t(g_A(P_i)) = E_i$. The distinguished eigenvector of $\pi_t(g_A(P_{n+1}))$ is a scalar multiple of Av_{n+1} which cannot lie in the plane spanned by $e_1, e_2, \ldots, e_{n-1}$ because v_{n+1} is not in the plane spanned by $v_1, v_2, \ldots, v_{n-1}$. Then $\pi_t(g_A(P_{n+1})) = (c_{ij})$ with $0 < c_{nn} < 1$. Let $B = (b_{ij}) \in \mathrm{GL}(n; \mathbf{R})$ be defined by

$$b_{ij} = 0 \quad \text{if } i \neq j; \qquad b_{ii} = \left[\frac{c_{nn}}{1 - c_{nn}}\right]^{1/2} \quad \text{if } i \leqslant n-1; \qquad b_{nn} = 1.$$

Then $\pi_t(g_B(E_i)) = E_i$ for $1 \leqslant i \leqslant n$. $(g_B \circ \pi_t \circ g_A)(P_{n+1})$ has as its diagonal entries

$$d_{ii} = \frac{c_{nn}}{1 - c_{nn}} c_{ii} \quad \text{for } 1 \leqslant i \leqslant n-1; \qquad d_{nn} = c_{nn}.$$

Then

$$\sum_{i=1}^{n-1} d_{ii} = \frac{c_{nn}}{1 - c_{nn}} \sum_{i=1}^{n-1} c_{ii} = \frac{c_{nn}}{1 - c_{nn}} (1 - c_{nn}) = c_{nn}$$

which equals d_{nn}. Therefore, $(\pi_t \circ g_B \circ \pi_t \circ g_A)(P_{n+1})$ has entry $\frac{1}{2}$ in the nnth place. The desired isometry is then $\pi_t \circ g_B \circ \pi_t \circ g_A$.

The next theorem gives a formula for the determinant of a straight simplex with vertices in $(X_t^n)_1$, in terms of the barycentric coordinates, which will be used in the estimates of §7.

THEOREM 5.14. *Let $P_1, P_2, \ldots, P_k$ be points in $(X_t^n)_1$ with $k \geqslant n$. Then*

$$\det\left(\sum_{i=1}^{k} t_i P_i\right) = \sum_{j_1 < j_2 < \cdots < j_n} \left(\prod_{i=1}^{n} t_{j_i}\right)\left[\mathrm{vol}(v_{j_1}, v_{j_2}, \ldots, v_{j_n})\right]^2.$$

PROOF. First consider the case $k = n$. Let w_i^j denote the jth column vector of P_i.

$$\det\left(\sum_{i=1}^{n} t_i P_i\right) = \sum_{i_1, i_2, \ldots, i_n = 1}^{n} \det\left[t_{i_1} w_{i_1}^1 \mid t_{i_2} w_{i_2}^2 \mid \cdots \mid t_{i_n} w_{i_n}^n\right].$$

In the case of rank 1 matrices, if two columns are taken from the same matrix the resulting determinant vanishes. The indicated sum then reduces to

$$\sum_{\tau \in S_n} \left(\prod_{i=1}^{n} t_i\right) \det\left[w_{\tau(1)}^1 \mid w_{\tau(2)}^2 \mid \cdots \mid w_{\tau(n)}^n\right].$$

Since each P_r is in $(X_t^n)_1$, $P_r = [(v_r \cdot e_i)(v_r \cdot e_j)]$ where v_r is the distinguished eigenvector. The sum then becomes

$$\left(\prod_{i=1}^{n} t_i\right)\left(\sum_{\tau\in S_n} \det\left[(v_{\tau(j)} \cdot e_i)(v_{\tau(j)} \cdot e_j)\right]\right)$$

$$= \left(\prod_{i=1}^{n} t_i\right)\left(\sum_{\tau\in S_n}\left[\prod_{j=1}^{n} (v_{\tau(j)} \cdot e_j)\det v_{\tau(j)} \cdot e_i\right]\right)$$

$$= \left(\prod_{i=1}^{n} t_i\right)\left(\sum_{\tau\in S_n} (-1)^{\tau}\left[\prod_{j=1}^{n} (v_{\tau(j)} \cdot e_j)\right]\det[v_j \cdot e_i]\right)$$

$$= \left(\prod_{i=1}^{n} t_i\right)\left(\det[v_j \cdot e_i]\right)^2$$

$$= \left(\prod_{i=1}^{n} t_i\right)\left[\operatorname{vol}(v_1, v_2, \ldots, v_n)\right]^2.$$

In the general case

$$\det\left(\sum_{i=1}^{k} t_i P_i\right) = \sum_{j_1<j_2<\cdots<j_n}\left(\det\left(\sum_{i=1}^{n} t_{j_i} P_{j_i}\right)\right)$$

$$= \sum_{j_1<j_2<\cdots<j_n}\left(\prod_{i=1}^{n} t_{j_i}\right)\left[\operatorname{vol}(v_{j_1}, v_{j_2}, \ldots, v_{j_n})\right]^2,$$

by the preceding calculation.

6. The Euclidean volume of slices of straight simplices with vertices in $(X_t^n)_1$. It must be shown that if σ is a straight simplex with vertices in $(X_t^n)_1$ then the volume of σ is bounded above by a function of n. By Theorem 4.3

$$\operatorname{vol}(\sigma) = \int_\sigma \frac{C_0(n)}{(\det X)^{(n+1)/2}} \bigwedge_{r\neq n} dx_r.$$

An upper bound on the volume of σ is found by dividing σ into slices and estimating the value of the integral on each slice by the product of the Euclidean volume of the slice and an estimate on the average value of the function $1/(\det X)^{(n+1)/2}$ over the slice. In the remainder of the paper $\operatorname{vol}_E(S)$ will denote the Euclidean volume of a set S. The notation $C_i(n)$ will be used to denote functions which depend only on n. The next theorem gives an estimate on the Euclidean volume of a straight simplex with vertices in $(X_t^n)_1$ in terms of the distinguished eigenvectors of some of the vertices.

THEOREM 6.1. *Let σ be a straight simplex with vertices $P_0, P_1, \ldots, P_{\delta(n)}$ in $(X_t^n)_1$ and suppose $P_0 = E_1$. Let $P_{\alpha_1}, P_{\alpha_2}, \ldots, P_{\alpha_n}$ be chosen such that $v_{\alpha_1}, v_{\alpha_2}, \ldots, v_{\alpha_n}$ have the largest vertical component (i.e. if $j \neq \alpha_i$ for any i then $v_j \cdot e_n \leq v_{\alpha_i} \cdot e_n$ for all i). Then*

$$\operatorname{vol}_E(\sigma) \leq C_1(n) \prod_{i=1}^{n} (v_{\alpha_i} \cdot e_n).$$

PROOF. First project σ from the hyperplane $(X_t^n)_1$ to the hyperplane $x_{nn} = 0$. For any straight simplex σ we have $\mathrm{vol}_E(\sigma) = k(n)\mathrm{vol}_E(p \circ \sigma)$ where p is the projection.

$$(6.2)\quad \mathrm{vol}_E(p \circ \sigma) = \left|\det\left[p(P_1 - P_0)\,|\,p(P_2 - P_0)\,|\cdots|\,p(P_{\delta(n)} - P_0)\right]\right|,$$

where $p(P_i - P_0)$ is the vector in $\mathbf{R}^{\delta(n)}$ determined by projecting the vector determined by $P_i - P_0$. Rewriting (6.2) in terms of the distinguished eigenvectors and taking the transpose we have

$$\mathrm{vol}_E(p \circ \sigma) = \left|\det\begin{array}{ccccc} 1 & 2 & & n-1 & n-\delta(n) \\ \left[(v_i \cdot e_1)^2 - 1\right. & |\ (v_i \cdot e_2)^2\ | & \cdots & |\ (v_i \cdot e_{n-1})^2\ | & \left. * \right] \end{array}\right|.$$

Adding the second through the $(n-1)$st column to the first we get

$$\mathrm{vol}_E(p \circ \sigma) = \left|\det\begin{array}{ccccc} 1 & 2 & & n-1 & n-\delta(n) \\ \left[-(v_i \cdot e_n)^2\right. & |\ (v_i \cdot e_2)^2\ | & \cdots & |\ (v_i \cdot e_{n-1})^2\ | & \left. * \right] \end{array}\right|.$$

In the columns n through $\delta(n)$ there are columns of the form $(v_i \cdot e_k)(v_i \cdot e_n)$ for each k such that $1 \leqslant k \leqslant n-1$. When the determinant is written out as a sum over all permutations τ, each term involves a factor of the form $\prod_{i=1}^n (v_{\beta_i} \cdot e_n)$ where the β_i are all different. Then

$$\mathrm{vol}_E(p \circ \sigma) \leqslant \sum_{\tau \in S_{\delta(n)}} \left(\prod_{i=1}^{n} (v_{\beta_{\tau(i)}} \cdot e_n)\right) |\text{products of other factors}|.$$

Since all entries in the matrix in (6.2) are bounded by 1 in absolute value the inequality becomes

$$\mathrm{vol}_E(p \circ \sigma) \leqslant \sum_{\tau \in S_{\delta(n)}} \left(\prod_{i=1}^{n} (v_{\beta_{\tau(i)}} \cdot e_n)\right).$$

By definition of the v_{α_i}, $\prod_{i=1}^n (v_{\beta_{\tau(i)}} \cdot e_n) \leqslant \prod_{i=1}^n (v_{\alpha_i} \cdot e_n)$ for each permutation. Then $\mathrm{vol}_E(\sigma) \leqslant C_1(n) \prod_{i=1}^n (v_{\alpha_i} \cdot e_n)$.

The next series of theorems prove an estimate on the Euclidean volume of slices of geometric simplices defined by $1 - 1/(m_i - 1) \leqslant u_i \leqslant 1 - 1/m_i$ for m_i a positive integer and $0 \leqslant i \leqslant k$ for some positive integer k. For example, if σ is a tetrahedron and $k = 1$ the slice is as in Figure 1. In the remainder of this section the vertices of a simplex, $P_0, P_1, \ldots, P_m$ are assumed to be points in some $\mathbf{R}^q$, but need not be in $\mathrm{Cl}(X_t^n)$.

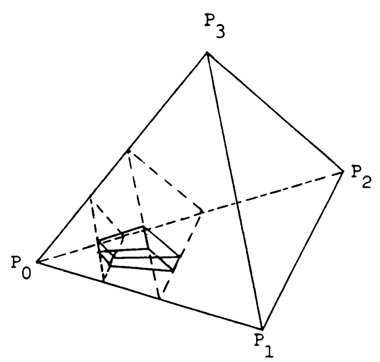

FIGURE 1

THEOREM 6.3. *Let σ be a geometric simplex with vertices $P_0, P_1, \ldots, P_{\delta(n)}$. Let k be a nonnegative integer such that $0 \leq k \leq \delta(n) - 1$ and choose α_k such that $0 \leq \alpha_k < 1$. Then the subset of σ defined by $\alpha_k \leq u_k \leq 1$ is the same as the geometric simplex σ' whose vertices Q_j are given by $Q_j = P_j$ for $j \leq k$ and $Q_j = \alpha_k P_k + (1 - \alpha_k)P_j$ for $j > k$.*

PROOF. Define $u_i' = u_i$ if $i \neq k$ and $u_k' = (u_k - \alpha_k)/(1 - \alpha_k)$. We have

$$1 - u_k' = \frac{1 - u_k}{1 - \alpha_k}. \tag{6.4}$$

The main point is that

$$\sum_{i=0}^{\delta(n)} u_i \left[\prod_{r<i} (1 - u_r)\right] P_i = \sum_{i=0}^{k} u_i' \left[\prod_{r<i} (1 - u_r')\right] P_i + \sum_{i=k+1}^{\delta(n)} u_i' \left[\prod_{r<i} (1 - u_r')\right] (\alpha_k P_k + (1 - \alpha_k)P_i). \tag{6.5}$$

It is clear that the coefficients of P_i are the same for $i < k$ on both sides of the equation since $u_i' = u_i$ if $i < k$. By (6.4) it is also immediate that the coefficients of P_i for $i > k$ are the same. The coefficient on the right-hand side of P_k is

$$\begin{aligned}
&\left[\prod_{r<k} (1 - u_r')\right]\left[u_k' + \alpha_k(1 - u_k') \sum_{i=k+1}^{\delta(n)} u_i' \prod_{r=k+1}^{i-1} (1 - u_r')\right] \\
&= \left[\prod_{r<k} (1 - u_r')\right]\left[u_k' + \alpha_k(1 - u_k')\right] \\
&= \left[\prod_{r<k} (1 - u_r)\right]\left[\frac{u_k - \alpha_k}{1 - \alpha_k} + \frac{\alpha_k(1 - u_k)}{1 - \alpha_k}\right] \\
&= u_k \prod_{r<k} (1 - u_r).
\end{aligned}$$

Hence, the coefficients of P_k are equal so (6.5) is established. Now if x is a point in the subset of σ defined by $\alpha_k \leq u_k \leq 1$ we have that $u_k' = (u_k - \alpha_k)/(1 - \alpha_k)$ must be between 0 and 1 and therefore x is in σ'. Conversely, if x is in σ' we have $u_k = u_k'(1 - \alpha_k) + \alpha_k$ so u_k is between α_k and 1. Then the theorem is established.

THEOREM 6.6. *Let σ be a geometric simplex with vertices $P_0, P_1, \ldots, P_{\delta(n)}$, and let k be an integer with $0 \leq k \leq \delta(n) - 1$. Then the subset σ' defined by $\alpha_k \leq u_k \leq 1$ has Euclidean volume satisfying*

$$\operatorname{vol}_E(\sigma') = (1 - \alpha_k)^{\delta(n)-k} \operatorname{vol}_E(\sigma).$$

PROOF. By Theorem 6.2, σ' is the simplex with vertices $Q_j = P_j$ for $j \leq k$ and $Q_j = \alpha_k P_k + (1 - \alpha_k)P_j$ for $j > k$. The face σ_k' spanned by $Q_k, Q_{k+1}, \ldots, Q_{\delta(n)}$ is similar to the face σ_k spanned by $P_k, P_{k+1}, \ldots, P_{\delta(n)}$ since for each i the distance from Q_k to Q_{k+1} is $(1 - \alpha_k) \cdot \operatorname{dist}(P_k, P_{k+1})$, and the solid angle with vertex at $Q_k = P_k$ is unchanged. Then

$$\operatorname{vol}_E(\sigma_k') = (1 - \alpha_k)^{\delta(n)-k} \operatorname{vol}_E(\sigma_k).$$

Then the volume of the simplex spanned by $Q_{k-1}, Q_k, \ldots, Q_{\delta(n)}$ is the product of $(1-\alpha_k)^{\delta(n)-k}$ and the volume of the simplex spanned by $P_{k-1}, P_k, \ldots, P_{\delta(n)}$ since the altitude from $Q_{k-1} = P_{k-1}$ to the base of the simplex is the same for each simplex. Continuing the process we get the desired result.

THEOREM 6.7. *The subset of σ defined by $\alpha_i \leqslant u_i \leqslant 1$ for all i such that $0 \leqslant i \leqslant k$ has Euclidean volume given by*

$$\left(\prod_{i=0}^{k}(1-\alpha_i)^{\delta(n)-i}\right)\operatorname{vol}_E(\sigma).$$

PROOF. Let σ_k be the simplex constructed in Theorem 6.3 which coincides with the subset of σ defined by $\alpha_k \leqslant u_k \leqslant 1$. By Theorem 6.6

$$\operatorname{vol}_E(\sigma_k) = (1-\alpha_k)^{\delta(n)-k}\operatorname{vol}_E(\sigma).$$

The subset of σ defined by $\alpha_k \leqslant u_k \leqslant 1$ and $\alpha_{k-1} \leqslant u_{k-1} \leqslant 1$ is the same as the subset of σ_k which has $\alpha_{k-1} = u'_{k-1} \leqslant 1$. Denote this simplex by σ_{k-1}. Applying Theorem 6.6 again we get

$$\begin{aligned}\operatorname{vol}_E(\sigma_{k-1}) &= (1-\alpha_{k-1})^{\delta(n)-(k-1)}\operatorname{vol}_E(\sigma_k)\\ &= (1-\alpha_{k-1})^{\delta(n)-(k-1)}(1-\alpha_k)^{\delta(n)-k}\operatorname{vol}_E(\sigma).\end{aligned}$$

Continuing this process yields the desired result.

It can be shown that if (M,μ) is a measure space and $B_i \subset A_i$ then

$$\mu\left(\bigcap_{i=0}^{k}(A_i - B_i)\right) = \sum_{\tau}(-1)^{|\tau^{-1}(1)|}\mu\left(\bigcap_{i=0}^{k} C_i^{\tau(i)}\right), \tag{6.8}$$

where $\tau\colon \{0,1,\ldots,k\} \to \{0,1\}$, $|\tau^{-1}(1)|$ is the cardinality of the set $\tau^{-1}(1)$, $C_i^0 = A_i$ and $C_i^1 = B_i$.

We are now ready to calculate the Euclidean volume of subsets of geometric simplices which are defined by the inequalities $\alpha_i \leqslant u_i \leqslant \beta_i$ for all i such that $0 \leqslant i \leqslant k$.

THEOREM 6.9. *Let S be the subset of a geometric simplex σ defined by $\alpha_i \leqslant u_i \leqslant \beta_i$ for all i such that $0 \leqslant i \leqslant k$. Then*

$$\operatorname{vol}_E(S) = \left\{\prod_{i=0}^{k}\left[(1-\alpha_i)^{\delta(n)-i} - (1-\beta_i)^{\delta(n)-i}\right]\right\}\operatorname{vol}_E(\sigma).$$

PROOF. Let A_i be the subset of σ defined by $\alpha_i \leqslant u_i \leqslant 1$ and let B_i be the subset of σ defined by $\beta_i \leqslant u_i \leqslant 1$. Then $S = \bigcap_{i=0}^{k}(A_i - B_i)$. Now applying (6.8) we get

$$\operatorname{vol}_E(S) = \sum_{\tau}(-1)^{|\tau^{-1}(1)|}\operatorname{vol}_E\left(\bigcap_{i=0}^{k} C_i^{\tau(i)}\right).$$

Let $\gamma_i^0 = \alpha_i$ and $\gamma_i^1 = \beta_i$. Then using Theorem 6.7 we get

$$\operatorname{vol}_E(S) = \sum_{\tau}(-1)^{|\tau^{-1}(1)|}\left(\prod_{i=0}^{k}\left(1-\gamma_i^{\tau(i)}\right)^{\delta(n)-i}\right)\operatorname{vol}_E(\sigma).$$

If $k=0$ this becomes $[(1-\alpha_0)^{\delta(n)}-(1-\beta_0)^{\delta(n)}]\mathrm{vol}_E(\sigma)$ so the formula holds. Now induct on k. Let $\tau' = \tau \mid \{0, 1, \ldots, k-1\}$. The coefficient of $(1-\alpha_k)^{\delta(n)-k}$ is

$$\sum_{\tau'} (-1)^{|\tau'^{-1}(1)|}\left[\prod_{i=0}^{k-1} \left(1-\gamma_i^{\tau'(i)}\right)^{\delta(n)-i}\right]\mathrm{vol}_E(\sigma),$$

while the coefficient of $(1-\beta_k)^{\delta(n)-k}$ is

$$\sum_{\tau'} (-1)^{|\tau'^{-1}(1)|+1}\left[\prod_{i=0}^{k-1} \left(1-\gamma_i^{\tau'(i)}\right)^{\delta(n)-i}\right]\mathrm{vol}_E(\sigma).$$

Then $\mathrm{vol}_E(S)$ is the product of $(1-\alpha_k)^{\delta(n)-k}-(1-\beta_k)^{\delta(n)-k}$ and

$$\left[\sum_{\tau'} (-1)^{|\tau'^{-1}(1)|}\prod_{i=0}^{k-1} \left(1-\gamma_i^{\tau'(i)}\right)^{\delta(n)-i}\right]\mathrm{vol}_E(\sigma).$$

Then by the inductive hypothesis we get the desired result.

THEOREM 6.10. *If* $\alpha_i = 1 - 1/(m_i - 1)$ *and* $\beta_i = 1 - 1/m_i$ *then*

$$\mathrm{vol}_E(S) \leqslant C_2(n)\left[\prod_{i=0}^{k} m_i^{i-1-\delta(n)}\right]\mathrm{vol}_E(\sigma).$$

PROOF. By Theorem 6.9

$$\mathrm{vol}_E(S) = \left\{\prod_{i=0}^{k}\left[(m_i-1)^{i-\delta(n)}-m_i^{i-\delta(n)}\right]\right\}\mathrm{vol}_E(\sigma)$$

$$= \left\{\prod_{i=0}^{k}\left[m_i^{\delta(n)-i}-(m_i-1)^{\delta(n)-i}\right]\left[(m_i-1)m_i\right]^{i-\delta(n)}\right\}\mathrm{vol}_E(\sigma).$$

In each factor, the numerator is a polynomial of degree $\delta(n)-i-1$ while each denominator is a polynomial of degree $2(\delta(n)-i)$. Each factor is then bounded above by the product of $m_i^{i-1-\delta(n)}$ and a constant depending only on i and n. Then

$$\mathrm{vol}_E(S) \leqslant C_2(n)\left[\prod_{i=0}^{k} m_i^{i-1-\delta(n)}\right]\mathrm{vol}_E(\sigma).$$

This result will be used in estimating the volume on slices in the next section.

7. The volume of a straight simplex with vertices in $(X_t^n)_1$. The volume of slices of a straight simplex σ will be written as the product of factors determined by the Euclidean volume of the slice and an integral over the corresponding part of the standard simplex which can be estimated. If the vertices of σ are given by $P_0, P_1, \ldots, P_{\delta(n)}$ where each P_i is in $(X_t^n)_1$, define a subset of the vertices $P_1', P_2', \ldots, P_{n+1}'$ inductively by letting P_i' be the first matrix in the ordering such that v_i' does not lie in the plane determined by $v_1', v_2', \ldots, v_{i-1}'$ for each i such that $1 \leqslant i \leqslant n$. Then let P_{n+1}' be any vertex different from P_n' such that v_{n+1}' does not lie in the plane determined by $v_1', v_2', \ldots, v_{n-1}'$. Note that Theorem 5.7 shows that if

(7.1) $\qquad P_i' = P_{\beta_i}$ then $\beta_i \leqslant \delta(i-1)+1$ if $1 \leqslant i \leqslant n$.

By Theorem 5.13 we may choose an isometry h taking P_i' to E_i if $1 \leqslant i \leqslant n$ and taking P_{n+1}' to a matrix whose entry $a_{nn} = \frac{1}{2}$. Since h is an isometry, $\mathrm{vol}(h \circ \sigma) = \mathrm{vol}(\sigma)$. The simplex spanned by $h(P_1')$, $h(P_2'), \ldots, h(P_{n-1}')$ lies in $(X_t^n)_{n-1}$. By Theorem 5.7 the maximum dimensional simplex which can lie in $(X_t^n)_{n-1}$ is $\delta(n-1)$. Equivalently there are at least

$$\delta(n) - \delta(n-1) = \left[\frac{n(n+1)}{2} - 1\right] - \left[\frac{(n-1)n}{2} - 1\right] = n \tag{7.2}$$

matrices whose distinguished eigenvectors do not lie in the hyperplane spanned by $e_1, e_2, \ldots, e_{n-1}$. Choose vertices $h(P_{\alpha_1})$, $h(P_{\alpha_2}), \ldots, h(P_{\alpha_n})$ of $h \circ \sigma$ such that the distinguished eigenvectors of the $h(P_{\alpha_i})$ have the largest vertical component. By (7.2) each of these vertical components is positive. In this section w_i will denote the distinguished eigenvector of $h(P_i)$ and w_i' will denote the distinguished eigenvector of $h(P_i')$.

By Theorem 6.1 if some w_{α_i} approaches the hyperplane spanned by $e_1, e_2, \ldots, e_{n-1}$ the Euclidean volume of $h \circ \sigma$ must approach 0. Simultaneously, the term in the formula for the determinant whose coefficient is $[\mathrm{vol}(w_1', w_2', \ldots, w_{n-1}', w_{\alpha_i})]^2$ approaches 0 since the parallelepiped is becoming degenerate. One of the main ideas considered in this section is the relationship of these two observations. This suggests using the estimate

$$\begin{aligned}
\det\left(\sum_{i=0}^{\delta(n)} t_i h(P_i)\right) &= \sum_{j_1<j_2<\cdots<j_n} \prod_{k=1}^{n} t_{j_k}\left[\mathrm{vol}\left(v_{j_1}, v_{j_2}, \ldots, v_{j_n}\right)\right]^2 \\
&\geqslant \sum_{i=1}^{n}\left(\prod_{k=1}^{n-1} t_k'\right) t_{\alpha_i}\left[\mathrm{vol}\left(w_1', w_2', \ldots, w_{n-1}', w_{\alpha_i}\right)\right]^2 \\
&= \sum_{i=1}^{n}\left(\prod_{k=1}^{n-1} t_k'\right) t_{\alpha_i}\left(w_{\alpha_i} \cdot e_n\right)^2.
\end{aligned} \tag{7.3}$$

THEOREM 7.4. *Let $P_0, P_1, \ldots, P_{\delta(n)}$ be points in $(X_t^n)_1$ which are vertices of a straight simplex σ. Let h be as in the preceding discussion and let S be the subset of $h \circ \sigma$ defined in Theorem 6.9. Let T be any subset of S and let Δ_T be the subset of the standard simplex corresponding to T. Then*

$$\mathrm{vol}(T) \leqslant C_5(n)\left[\prod_{i=1}^{n}\left(w_{\alpha_i} \cdot e_n\right)\right] \int_{\Delta_T} \frac{dt_1 dt_2 \cdots dt_{\delta(n)}}{\left[\sum_{i=1}^{n}\left(\prod_{k=1}^{n-1} t_k'\right) t_{\alpha_i}\left(w_{\alpha_i} \cdot e_n\right)^2\right]^{(n+1)/2}}.$$

PROOF. Define a map f from the standard simplex Δ to $h(\sigma)$ by $f(t_0, t_1, \ldots, t_{\delta(n)}) = \sum_i t_i h(P_i)$. By the change of variables formula we have

$$\begin{aligned}
\mathrm{vol}(T) &= \int_T \frac{C_0(n)}{(\det X)^{(n+1)/2}} \bigwedge_{r \neq n} dx_r \\
&= \int_{\Delta_T} \frac{C_0(n)\,|\det f'|\, dt_1 dt_2 \cdots dt_{\delta(n)}}{\left[\sum_{j_1<j_2<\cdots<j_n}\left(\prod_{k=1}^{n} t_{j_k}\right)\left[\mathrm{vol}\left(w_{j_1}, w_{j_2}, \ldots, w_{j_n}\right)\right]^2\right]^{(n+1)/2}}.
\end{aligned} \tag{7.5}$$

Also, since $\int_{h(\sigma)} \bigwedge_{r\neq n} dx_r = \int_\Delta |\det f'|\, dt_1 dt_2 \cdots dt_{\delta(n)}$ we get

$$|\det f'| = \frac{\mathrm{vol}_E(p \circ h(\sigma))}{\mathrm{vol}_E(\Delta)} = C_4(n)\mathrm{vol}_E(h(\sigma)), \tag{7.6}$$

where p is as in the proof of Theorem 6.1. Plugging (7.3) and (7.6) into (7.5) we have

$$\mathrm{vol}(T) \leqslant C_0(n)C_4(n)\mathrm{vol}_E(h(\sigma))\int_{\Delta_T} \frac{dt_1 dt_2 \cdots dt_{\delta(n)}}{\left[\sum_{i=1}^n\left(\prod_{k=1}^{n-1}t'_k\right)t_{\alpha_i}(w_{\alpha_i}\cdot e_n)^2\right]^{(n+1)/2}}.$$

By Theorem 6.1, $\mathrm{vol}_E(h(\sigma)) \leqslant C_1(n)\prod_{i=1}^n(w_{\alpha_i}\cdot e_n)$. Hence the result follows.

In the remainder of the paper let S be the set defined by $1 - 1/(m_i - 1) \leqslant u_i \leqslant 1 - 1/m_i$ for all i such that $0 \leqslant i \leqslant \delta(n-1)$ (i.e., in Theorem 6.9 take $k = \delta(n-1)$). If x is a point in S we can write

$$\begin{aligned} x = &\sum_{i=0}^{\delta(n-1)} u_i\left[\prod_{k<i}(1-u_k)\right]P_i \\ &+\left(\prod_{k\leqslant\delta(n-1)}(1-u_k)\right)\sum_{i=\delta(n-1)+1}^{\delta(n)} u_i\left[\prod_{k=\delta(n-1)+1}^{i-1}(1-u_k)\right]P_i. \end{aligned}$$

Then the coefficients of $P_0, P_1, \ldots, P_{\delta(n-1)}$ and

$$\sum_{i=\delta(n-1)+1}^{\delta(n)} u_i\left[\prod_{k=\delta(n-1)+1}^{i-1}(1-u_k)\right]P_i$$

are all between 0 and 1. Hence, x must be an interior point of this $\delta(n-1)+1$ simplex. By Theorem 5.7 this simplex cannot lie entirely on the boundary of X_t^n and therefore x is an interior point of $\mathrm{Cl}(X_t^n)$. Let $A(\Delta_T)$ denote the average value of the function

$$\frac{1}{\left[\sum_{i=1}^n\left(\prod_{k=1}^{n-1}t'_k\right)t_{\alpha_i}(w_{\alpha_i}\cdot e_n)^2\right]^{(n+1)/2}}$$

over the set Δ_T.

THEOREM 7.7 $\mathrm{vol}(T) \leqslant C_6(n)\prod_{i=1}^n(w_{\alpha_i}\cdot e_n)\prod_{i=0}^{\delta(n-1)} m_i^{i-1-\delta(n)}A(\Delta_T)$.

PROOF.

$$\int_{\Delta_T}\frac{dt_1 dt_2\cdots dt_{\delta(n)}}{\left[\sum_{i=1}^n\left(\prod_{k=1}^{n-1}t'_k\right)t_{\alpha_i}(w_{\alpha_i}\cdot e_n)^2\right]^{(n+1)/2}} = A(\Delta_T)\mathrm{vol}_E(\Delta_T).$$

By Theorem 6.10

$$\mathrm{vol}_E(\Delta_T) \leqslant \mathrm{vol}_E(\Delta_S) \leqslant C_2(n)\prod_{i=0}^{\delta(n-1)} m_i^{i-1-\delta(n)}\mathrm{vol}_E(\Delta). \tag{7.8}$$

Then combining (7.8) with Theorem 7.4 we have

$$(7.9)\quad \operatorname{vol}(T) \leq C_5(n) \prod_{i=1}^{n} (w_{\alpha_i} \cdot e_n) C_2(n) \prod_{i=0}^{\delta(n-1)} m_i^{i-1-\delta(n)} \operatorname{vol}_E(\Delta) A(\Delta_T)$$
$$= C_6(n) \prod_{i=1}^{n} (w_{\alpha_i} \cdot e_n) \prod_{i=0}^{\delta(n-1)} m_i^{i-1-\delta(n)} A(\Delta_T).$$

Now let T be the intersection of S with the subset of σ defined by $t_i \geq t_{i+1}$ for all i. An estimate on $A(\Delta_T)$ is needed. Let $\Delta_T((\gamma_1, c_1), (\gamma_2, c_2), \ldots, (\gamma_r, c_r))$ be a subset of Δ_T defined by the barycentric coordinates $t_{\gamma_1}, t_{\gamma_2}, \ldots, t_{\gamma_r}$ being constants $c_1, c_2, \ldots, c_r$. If $t_{\gamma_1}, t_{\gamma_2}, \ldots, t_{\gamma_r}$ are allowed to range over all possible constants, it is clear that

$$A(\Delta_T) \leq \sup_{c_1, c_2, \ldots, c_r} \left(A(\Delta_T((\gamma_1, c_1), (\gamma_2, c_2), \ldots, (\gamma_r, c_r))) \right).$$

THEOREM 7.10.

$$A(\Delta_T) \leq C_8(n) \left[\left(\prod_{k=0}^{\delta(n-1)} m_k \right) \left(\prod_{i=1}^{n-2} \left(\prod_{k=1}^{i} m_{\delta(i-1)+k} \right)^{n-i-1} \right) \right]^{(n+1)/2} I(\sigma),$$

where

$$I(\sigma) = \int_{\Delta} \frac{dy_1 dy_2 \cdots dy_{n-1}}{\left[\sum_{i=1}^{n} y_i (w_{\alpha_i} \cdot e_n)^2 \right]^{(n+1)/2}}$$

and $y_n = 1 - \sum_{i=1}^{n-1} y_i$.

PROOF. Let $\gamma_1, \gamma_2, \ldots, \gamma_r$ be the set of indices which consist of all integers j such that $0 \leq j \leq \delta(n)$ and $j \neq \alpha_i$ for any i. Let the constants $c_1, c_2, \ldots, c_r$ be any possible values in Δ_T. Then on $\Delta_T((\gamma_1, c_1), (\gamma_2, c_2), \ldots, (\gamma_r, c_r))$ which will hereafter be denoted by $\Delta_T(\Gamma, c)$, we have

$$\sum_{i=1}^{n} t_{\alpha_i} = \varepsilon \quad \text{where } \varepsilon = 1 - \sum_i c_i.$$

Let Δ_ε denote the geometric simplex in $\mathbf{R}^{n-1}$ whose vertices are the origin and the points a distance ε from the origin along each coordinate axis. Then

$$(7.11)\quad A(\Delta_T(\Gamma, c)) = \frac{1}{\operatorname{vol}_E(\Delta_\varepsilon)} \int_{\Delta_\varepsilon} \frac{dt_{\alpha_1} dt_{\alpha_2} \cdots dt_{\alpha_{n-1}}}{\left[\sum_{i=1}^{n} \left(\prod_{k=1}^{n-1} t'_k \right) t_{\alpha_i} (w_{\alpha_i} \cdot e_n)^2 \right]^{(n+1)/2}}$$
$$= \frac{C_7(n)}{\varepsilon^{n-1} \left(\prod_{k=1}^{n-1} t'_k \right)} \int_{\Delta_\varepsilon} \frac{dt_{\alpha_1} dt_{\alpha_2} \cdots dt_{\alpha_{n-1}}}{\left[\sum_{i=1}^{n} t_{\alpha_i} (w_{\alpha_i} \cdot e_n)^2 \right]^{(n+1)/2}}.$$

This last equation holds since $t'_1, t'_2, \ldots, t'_{n-1}$ are all constant on $\Delta_T(\Gamma, c)$. Now define a function $f\colon \Delta \to \Delta_\varepsilon$ by

$$f(y_1 y_2, \ldots, y_{n-1}) = (\varepsilon y_1, \varepsilon y_2, \ldots, \varepsilon y_{n-1}).$$

By the change of variables formula (7.11) becomes

$$(7.12)\qquad A(\Delta_T(\Gamma,c)) = \frac{C_7(n)}{\varepsilon^{n-1}\left(\prod_{k=1}^{n-1} t'_k\right)} \int_\Delta \frac{\varepsilon^{n-1} dy_1 dy_2 \cdots dy_{n-1}}{\left[\sum_{i=1}^n \varepsilon y_i (w_{\alpha_i}\cdot e_n)^2\right]^{(n+1)/2}}$$

$$= \frac{C_7(n)}{\varepsilon^{(n+1)/2}\left(\prod_{k=1}^{n-1} t'_k\right)} \int_\Delta \frac{dy_1 dy_2 \cdots dy_{n-1}}{\left[\sum_{i=1}^n y_i (w_{\alpha_i}\cdot e_n)^2\right]^{(n+1)/2}}.$$

Now write

$$t'_i = t_{\beta_i} = u_{\beta_i} \prod_{k<\beta_i} (1-u_k) \geqslant \frac{1}{\delta(n)+1} \prod_{k<\beta_i} (1-u_k)$$

by Theorem 5.6. By (7.1) we have $\beta_i \leqslant \delta(i-1)+1$ so

$$t'_i \geqslant \frac{1}{\delta(n)+1} \prod_{k\leqslant\delta(i-1)} (1-u_k).$$

Then

$$\prod_{k=1}^{n-1} t'_k \geqslant (\delta(n)+1)^{1-n} \prod_{i=1}^{n-2} \left(\prod_{k=\delta(i-1)+1}^{\delta(i)} (1-u_k) \right)^{n-i-1}.$$

On Δ_S each u_i for $0 \leqslant i \leqslant \delta(n-1)$ satisfies $u_k \leqslant 1 - 1/m_i$ hence

$$\prod_{k=1}^{n-1} t'_k \geqslant (\delta(n)+1)^{1-n} \prod_{i=1}^{n-2} \left(\prod_{k=\delta(i-1)+1}^{\delta(i)} \frac{1}{m_k} \right)^{n-i-1}.$$

Therefore

$$(7.13)\qquad \frac{1}{\left(\prod_{k=1}^{n-1} t_k\right)^{(n+1)/2}} \leqslant \left[(\delta(n)+1)^{n-1}\right]^{(n+1)/2}$$

$$\times \left[\prod_{i=1}^{n-2} \left(\prod_{k=\delta(i-1)+1}^{\delta(i)} m_k \right)^{n-i-1} \right]^{(n+1)/2}.$$

Also, using the assumption that $t_{i+1} \geqslant t_i$ for all i we have

$$\varepsilon = \sum_{i=1}^n t_{\alpha_i} \geqslant \sum_{i=1}^n t_{\delta(n)-i+1}$$

$$= \left(\prod_{k\leqslant\delta(n-1)} (1-u_k) \right) \sum_{j=1}^n u_{\delta(n-1)+j} \prod_{k=1}^{j-1} (1-u_{\delta(n-1)+k})$$

$$= \prod_{k\leqslant\delta(n-1)} (1-u_k) = \prod_{k\leqslant\delta(n-1)} \frac{1}{m_k}.$$

Then

$$(7.14)\qquad \frac{1}{\varepsilon^{(n+1)/2}} \leqslant \left[\prod_{k\leqslant\delta(n-1)} m_k \right]^{(n+1)/2}.$$

Then combining (7.12), (7.13), and (7.14) we have the desired result.

A picture of a lower dimensional analog to the sets $\Delta_T(\Gamma, c)$ can be given. In Figure 1 think of P_2 and P_3 as being the set of the P_{α_i}; and the box drawn in the figure as Δ_T. Then the sets $\Delta_T(\Gamma, c)$ are line segments formed by intersecting lines parallel to the line determined by P_2 and P_3 with the box.

Combining Theorem 7.10 with (7.9) we get

$$\text{vol}(T) \leqslant C_9(n) f(m_0, m_1, \ldots, m_{\delta(n-1)}) \prod_{i=1}^{n} (w_{\alpha_i} \cdot e_n) I(\sigma), \tag{7.15}$$

where

$$f(m_0, m_1, \ldots, m_{\delta(n-1)})$$
$$= \prod_{i=0}^{\delta(n-1)} m_i^{i-1-\delta(n)} \left[\left(\prod_{k=0}^{\delta(n-1)} m_k \right) \left(\prod_{i=1}^{n-2} \left(\prod_{k=1}^{i} m_{\delta(i-1)+k} \right)^{n-i-1} \right) \right]^{(n+1)/2}.$$

The next two theorems show that $\prod_{i=1}^{n}(w_{\alpha_i} \cdot e_n) I(\sigma)$ in (7.15) is bounded by a function which depends only on n.

THEOREM 7.16.

$$\prod_{i=1}^{n} (w_{\alpha_i} \cdot e_n) I(\sigma) \leqslant C_{10}(n) \int_\Delta \frac{dy_1 dy_2 \cdots dy_{n-1}}{\left(\prod_{i=1}^{n} y_i\right)^{3/4}}.$$

PROOF. The integral can be written as

$$\int_\Delta \frac{dy_1 dy_2 \cdots dy_{n-1}}{\left[\sum_{i=1}^{n} y_i (w_{\alpha_i} \cdot e_n)^2\right]^{n/2} \left[\sum_{i=1}^{n} y_i (w_{\alpha_i} \cdot e_n)^2\right]^{1/2}}.$$

Without loss of generality we may assume that w_{α_1} and w_{α_2} have the largest vertical components. Using the inequality between arithmetic and geometric means in the first factor in the denominator and dropping all but the first two terms in the second factor we have that the integral is bounded above by

$$\int_\Delta \frac{1}{n^{n/2}} \frac{dy_1 dy_2 \cdots dy_{n-1}}{\left(\prod_{i=1}^{n} y_i\right)^{1/2} \prod_{i=1}^{n} (w_{\alpha_i} \cdot e_n) \left[y_1 (w_{\alpha_1} \cdot e_n)^2 + y_2 (w_{\alpha_2} \cdot e_n)^2 \right]^{1/2}}. \tag{7.17}$$

Using the inequality between the arithmetic and geometric means in the last factor of the denominator shows that $\prod_{i=1}^{n}(w_{\alpha_i} \cdot e_n) I(\sigma)$ is bounded above by

$$\int_\Delta \frac{1}{n^{n/2}} \frac{dy_1 dy_2 \cdots dy_{n-1}}{y_1^{3/4} y_2^{3/4} \left(\prod_{i=3}^{n} y_i\right)^{1/2}} \leqslant \int_\Delta \frac{C_{10}(n) dy_1 dy_2 \cdots dy_{n-1}}{\left(\prod_{i=1}^{n} y_i\right)^{3/4}}.$$

THEOREM 7.18. *$\int_\Delta dy_1 dy_2 \cdots dy_{n-1} / \left(\prod_{i=1}^{n} y_i\right)^{3/4}$ converges.*

PROOF. Note that if $0 \leqslant a \leqslant 1$

$$\int_0^a \frac{dx}{[x(a-x)]^{3/4}} = 2 \int_0^{a/2} \frac{dx}{[x(a-x)]^{3/4}} \tag{7.19}$$
$$\leqslant \frac{2}{(a/2)^{3/4}} \int_0^{a/2} \frac{dx}{x^{3/4}} \leqslant \frac{8 \cdot 2^{3/4}}{a^{3/4}}.$$

The proof is by induction on n. If $n = 1$ the integral is just the integral in (7.19) where $a = 1$. In general we can rewrite the integral as

$$(7.20)\quad \int_{\Delta_{n-2}} \frac{1}{\left(\prod_{i=1}^{n-2} y_i\right)^{3/4}} \int_0^{1-\sum_{i=1}^{n-2} y_i} \frac{dy_{n-1}}{\left[y_{n-1}\left(1 - \sum_{i=1}^{n-1} y_i\right)\right]^{3/4}} dy_1\, dy_2 \cdots dy_{n-2}.$$

Now set $a = 1 - \sum_{i=1}^{n-2} y_i$ so that (7.20) may be rewritten as

$$\int_{\Delta_{n-2}} \frac{1}{\left(\prod_{i=1}^{n-2} y_i\right)^{3/4}} \int_0^{a} \frac{dy_{n-1}}{\left[y_{n-1}(a - y_{n-1})\right]^{3/4}} dy_1\, dy_2 \cdots dy_{n-2}$$
$$\leqslant 8 \cdot 2^{3/4} \int_{\Delta_{n-2}} \frac{dy_1 dy_2 \cdots dy_{n-2}}{\left(\prod_{i=1}^{n-2} y_i\right)^{3/4} \left(1 - \sum_{i=1}^{n-2} y_i\right)^{3/4}}$$

by (7.19). Then by the inductive hypothesis the theorem is established.

We are now ready to prove the main result of this section. It will be shown that the volume of the subset U defined by $t_i \geqslant t_{i+1}$ for any straight $\delta(n)$ simplex with vertices in $(X_t^n)_1$ is bounded above by a function of n. Combining the results of Theorems 7.16 and 7.18 with (7.15) we already have that

$$\operatorname{vol}(T) \leqslant C_{12}(n) f\left(m_0, m_1, \ldots, m_{\delta(n-1)}\right).$$

Hence the volume of each slice T is bounded above by a function which is independent of the simplex σ. To see that the volume of U is bounded above by a function of n, the estimates on the volumes of the slices will be summed and shown to be finite. An analogous situation can be seen in three dimensions by again referring to Figure 1. Consider the vertices P_0 and P_1 as taking the part of the vertices $P_0, P_1, \ldots, P_{\delta(n-1)}$ and let P_2 and P_3 represent the remaining vertices. As shown before, on the slices actually being considered the determinant is nonvanishing. If we suppose the determinant to be zero on the simplex spanned by P_0 and P_1, and nonzero elsewhere, we get the analogous picture. The determinant must be nonzero everywhere on the slice in Figure 1. Notice that as m_1 approaches infinity the slices drawn would approach the line segment determined by P_0 and P_1. The determinant would approach zero while the Euclidean volume would also approach zero. Similarly, as m_0 approaches infinity the slices would become small and approach P_0. We have the following theorem.

THEOREM 7.21. *If σ is any straight $\delta(n)$ simplex with vertices in $(X_t^n)_1$ the subset U of σ has volume which is bounded above by a function $C(n)$ of n.*

PROOF.

$$\operatorname{vol}(U) \leqslant C_{12}\left(\sum_{m_0=1}^{\infty} m_0^{p_0}\right)\left(\sum_{m_1=1}^{\infty} m_1^{p_1}\right) \cdots \left(\sum_{m_{\delta(n-1)}=1}^{\infty} m_{\delta(n-1)}^{p_{\delta(n-1)}}\right),$$

where p_i denotes the exponent of the corresponding m_i. From (7.15) it is clear that the largest possible exponents p_i occur when $i = \delta(j)$ for some j. The exponent on $m_{\delta(j)}$ is given by

$$\begin{aligned} p_{\delta(j)} &= \frac{n+1}{2} + (n-j-1)\frac{n+1}{2} - [\delta(n) - \delta(j) + 1] \\ &= \frac{j}{2}(j-n) - 1. \end{aligned}$$

Since all possible values for j are less than or equal to $n-1$, the exponents $p_{\delta(j)}$ are bounded above by $-\frac{3}{2}$ and therefore each series converges and $\operatorname{vol}(U)$ is bounded above by a function of n.

By Theorem 5.4 the volume of every straight simplex with vertices in $(X_t^n)_1$ is then bounded by a function of n. Then by (3.5), $\|[M]\| \neq 0$ if $M = X_d^n/\Gamma$.

8. Other results. Let M and N be closed, oriented manifolds whose universal cover is X_d^n. If $f\colon M \to N$ is a continuous map (2.5) shows that

$$|\deg f| \leqslant \frac{\|[M]\|}{\|[N]\|}$$

since we have shown that $\|[N]\| \neq 0$. By (2.13) this can be rewritten as

$$|\deg f| \leqslant \frac{C\operatorname{vol}(M)}{C\operatorname{vol}(N)} = \frac{\operatorname{vol}(M)}{\operatorname{vol}(N)}. \tag{8.1}$$

This result is well known for $n = 2$. For $n \geqslant 3$ it can also be derived as a consequence of the rigidity theorems of Margulis and Mostow.

The result of this paper combined with results of Trauber, cf. **[4]**, and Hirsch and Thurston in **[5]** give a proof that if Γ is a subgroup of $SL(n; \mathbf{R})$ such that X_d^n/Γ is a closed, oriented manifold, then the growth of Γ is exponential. Trauber showed that if $\pi_1(M)$ is amenable then $\|[M]\| = 0$. Hirsch and Thurston showed that if a group has subexponential growth then it is amenable. In this paper it has been shown that $\|[M]\| \neq 0$ if $M = X_d^n/\Gamma$ hence Γ has exponential growth. This result also follows from Tits' theorem, cf. **[9]**.

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